On the Game of Googol

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Abstract: In the classical secretary problem the decision maker can only observe the relative ranks of the items presented. Recently, Ferguson – building on ideas of Stewart – showed that, in a game theoretic sense, there is no advantage if the actual values of the random variables underlying the relative ranks can be observed (game of googol). We extend this to the case where the number of items is unknown with a known upper bound. Corollary 3 extends one of the main results in [HK] to *all* randomized stopping times. We also include a modified, somewhat more formal argument for Ferguson's result.

1 Introduction

In an entertaining yet serious article, T. Ferguson [F] recently pursued the question: Who solved the secretary problem? The first published version of the problem was described as the (two-person zero-sum) game of googol in Martin Gardner's February 1960 column of the *Scientific American.* Player I chooses n numbers and presents them one by one in random order to Player II, who may stop whenever he pleases, and wins if the last number observed is the largest of all the numbers. In contrast to the usual treatment of the secretary problem, Player II can see the actual numbers presented to him and not just their relative ranks.

It follows from results of Samuels [S] that the lower value of the game of googol is equal to the value φ_n that can be obtained by considering only the relative ranks. On the other hand, Ferguson showed that, for any $\varepsilon > 0$, Player I has a strategy for choosing the numbers $X_1, \ldots, X_n$ which guarantees that Player II cannot succeed with probability larger than $\varphi_n + \varepsilon$. Thus, the value of the game of googol exists and equals φ_n. Ferguson used a class of distributions for $X_1, \ldots, X_n$, already employed by Berezovskiy and Gnedin [BG] to derive a result similar to that of Samuels.

In a recent article [HK] the present authors studied the secretary problem when the number of items presented is a random variable N with an unknown distribution but with known upper bound n. The least favorable distribution for N and the (ran-

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domized) minimax optimal strategy for Player II observing only the relative ranks were determined. In the present paper it is shown that the advantage which Player II might have from using the X_i instead of the ranks is, again, arbitrarily small, if Player I uses the distributions proposed by Berezovskiy and Gnedin for the X_i, and, independently, the least favorable distribution of N.

2 Fixed Number of Observations

The main purpose of this section is to record some known results on the secretary problem and the game of googol and some modifications, simplifications and extensions (to include ties and randomized stopping). The present variants will be needed in § 3.

Let us start with a discussion of randomized stopping times. If $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \dots$ is an increasing sequence of σ-algebras in a probability space $(\Omega, \mathscr{F}, P)$, $\mathscr{F}_\infty$ shall denote the σ-algebra generated by the union of the $\mathscr{F}_n$'s. By enlarging the underlying probability space, if necessary, it may be assumed that there exists a sequence of independent random variables $U_1, U_2, \dots$ independent of $\mathscr{F}_\infty$, and having uniform distribution in $[0, 1]$.

Following [PS], we call a random variable t with values in $\{1, 2, \dots\} \cup \{\infty\}$ a *randomized stopping time* of $(\mathscr{F}_n)$ if for each $m = 1, 2, \dots$ the event $\{t > m\}$ and the σ-algebra $\mathscr{F}_\infty$ are conditionally independent given $\mathscr{F}_m$. $\mathscr{T}_r((\mathscr{F}_m))$ denotes the family of randomized stopping times for $(\mathscr{F}_m)$, and $\mathscr{T}((\mathscr{F}_m))$ denotes the family of stopping times for $(\mathscr{F}_m)$.

If $q = (q_1, q_2, \dots)$ is a sequence of $[0, 1]$-values random variables such that q_m is $\mathscr{F}_m$-measurable for each m, a randomized stopping t_q is given by

$$t_q = \inf\{m \geq 1 : U_m \leq q_m\}.$$

We have

$$P(t_q > m \mid \mathscr{F}_\infty) = P(t_q > m \mid \mathscr{F}_m) = (1 - q_1)(1 - q_2) \dots (1 - q_m).$$

Pitman and Speed [PS] have sketched an argument showing that (for large enough probability spaces) any randomized stopping time t can be represented "isomorphically" as above: if $t \in \mathscr{T}_r(\mathscr{F}_m))$ is given, put (taking $0/0 = 1$)

$$q_m = P(t = m \mid \mathscr{F}_m) / P(t \geq m \mid \mathscr{F}_m). \tag{2.1}$$

Then the randomized stopping times t_q and t have the same conditional distribution given $\mathscr{F}_\infty$. Hence, we can replace t by t_q in any stopping problem with $\mathscr{F}_\infty$-measurable reward Z_m for stopping at time m without changing the expected reward.

For finite sequences $\mathscr{F}_1 \subset \dots \subset \mathscr{F}_n$, these definitions and remarks must be modified in the obvious way; for example, $\mathscr{F}_n$ replaces $\mathscr{F}_\infty$.

For a finite sequence $X = (X_1, \dots, X_n)$ of random variables taking values in a totally ordered set, and $1 \leq i \leq m$, let

$$R_{i,m}:=R_{i,m}(X):=1+\operatorname{card}\{j\leq m: X_j>X_i\}$$

denote the relative rank of X_i in $X_1,\ldots,X_m$; (card A is the number of elements of A).

If $\mathscr{F}_m$ is the σ-algebra generated by $X_1,\ldots,X_m$, we write $\mathscr{T}$ (or $\mathscr{T}(X)$) for $\mathscr{T}((\mathscr{F}_m))$ and $\mathscr{T}_r$ (or $\mathscr{T}_r(X)$) for $\mathscr{T}_r((\mathscr{F}_m))$. If $\mathscr{G}_m$ is the σ-algebra generated by $R_{1,m},\ldots,R_{m,m}$, then $\mathscr{G}_m\subset\mathscr{G}_{m+1}$. We write $\mathscr{T}(R)$ or $\mathscr{T}(R(X))$ for $\mathscr{T}((\mathscr{G}_m))$ and $\mathscr{T}_r(R)$ or $\mathscr{T}_r(R(X))$ for $\mathscr{T}_r((\mathscr{G}_m))$. Note that $\mathscr{G}_m\subset\mathscr{F}_m$, and hence $\mathscr{T}(R)\subset\mathscr{T}(X)$ and $\mathscr{T}_r(R)\subset\mathscr{T}_r(X)$.

We shall also need the random variables $M_i=M_i(X)=\max(X_1,\ldots,X_i)$.

In the classical formulation of the secretary problem (see Ferguson [F]), it is usually assumed that $X_1,\ldots,X_n$ is a random permutation of a set of n *distinct* real numbers, that is, $X_1,\ldots,X_n$ are exchangeable with

$$P(X_i=X_j \text{ for some } i\neq j)=0.$$

The objective is to find a stopping time $t\in\mathscr{T}(R)$ based only on the relative ranks for which $P(X_t=M_n)$ is as large as possible. In this case (without ties) the optimal stopping rule $\hat{t}_n$ is well known and is obtained as follows.

Set $s_0=0$ and for $j\geq 1$, $s_j=\sum_{i=1}^{j} i^{-1}$, and let k_n be the nonnegative integer satisfying

$$s_{n-1}-s_{k_n-1}\geq 1>s_{n-1}-s_{k_n}. \tag{2.2}$$

Then the stopping time $\hat{t}_n$ given by

$$\begin{aligned}\hat{t}_n&=\min\{n,\ \min\{j>k_n: X_j\geq X_i \text{ for all } i<j\}\}\\&=\min\{n,\ \min\{j>k_n: R_{j,j}=1\}\}\end{aligned} \tag{2.3}$$

is optimal, even over the class $\mathscr{T}_r(R)$ of randomized stopping times based on the ranks, that is

$$P(X_{\hat{t}_n}=M_n)=\sup_{t\in\mathscr{T}_r(R)} P(X_t=M_n). \tag{2.4}$$

As long as there are no ties, the value in (2.4) does not depend on the (exchangeable) distribution of $X_1,\ldots,X_n$, and, as in Ferguson [F], will be denoted by φ_n.

The next proposition extends the minimax version of (2.4) slightly to include the possibility of ties. It is vaguely hinted at, although not explicitly stated or proved, in Campbell [C]. Let $\mathscr{E}$ denote the family of all exchangeable sequences $X=(X_1,\ldots,X_n)$.

Proposition 2.1: There exists a randomized stopping rule $\hat{t}'_n\in\mathscr{T}_r(R)$ such that

$$P(X_{\hat{t}'_n}=M_n(X))\geq\varphi_n \tag{2.5}$$

holds for all exchangeable sequences $X_1,\ldots,X_n$. Hence

$$\inf_{X\in\mathscr{E}}\ \sup_{t\in\mathscr{T}_r(R)} P(X_t=M_n)=\varphi_n=\sup_{t\in\mathscr{T}_r(R)}\ \inf_{X\in\mathscr{E}} P(X_t=M_n).$$

Proof: Let $X_1, \ldots, X_n$ be exchangeable. Enlarging the probability space, if necessary, let $U_1, U_2, \ldots, U_n$ be iid random variables with uniform distribution in $[0, 1]$, independent of $X_1, \ldots, X_n$. Put $X_i'=(X_i, U_i)$, and write $X_i' \prec X_j'$ if either $X_i<X_j$, or $X_i=X_j$ and $U_i<U_j$. The process $X'=(X_1', X_2', \ldots, X_n')$ is exchangeable and almost surely has no ties. The stopping time

$$\hat{t}_n'=\min\{n, \min\{j>k_n: R_{i,j}(X')=1\}\}$$

is a randomized stopping time based on the ranks of the X_j's, that is, it belongs to $\mathscr{T}_r(R(X))$. It does not depend on the distribution of $X_1, \ldots, X_n$. If $X_i'(\omega)$ is maximal among $X_1'(\omega), \ldots, X_n'(\omega)$, then $X_i(\omega)=M_n(X)(\omega)$, so (2.5) holds. The second assertion follows from the first by using (2.4) and considering sequences $X_1, \ldots, X_n$ which have almost surely no ties. □

The next objective will be to apply ideas of Stewart [St], Berezovskiy, Gnedin, and Ferguson to show that even if Player II can use stopping times based on the actual values of the process, as opposed to just their relative ranks, in general he can still not improve his probability φ_n of winning by more than ε.

Let $I(A)$ denote the indicator function of a set A. $U(a, b)$ denotes the uniform distribution in the interval $[a, b]$. Recall that the Pareto distribution with parameters $\alpha>0$ and $m>0$ is the distribution on $\mathbb{R}$ with density

$$g(\theta \mid \alpha, m)=\alpha m^\alpha \theta^{-(\alpha+1)} I(\theta>m).$$

Definition 2.2: An $(n+1)$-dimensional random vector $(\Theta, X_1, \ldots, X_n)$ is said to be Pa$U(\alpha, m, n)$-distributed if: (i) Θ has a Pareto distribution with parameters $\alpha>0$, $m>0$ and (ii) given $\Theta=\theta$, the random variables $X_1, \ldots, X_n$ are iid $U(0, \theta)$.

Lemma 2.3: If $(\Theta, X_1, \ldots, X_n)$ is PaU(α, m, n), then (i) $X_1, X_2, \ldots, X_n$ are exchangeable with $P(X_i=X_j$ for some $i\neq j)=0$, and (ii) the conditional distribution of $(\Theta, X_{j+1}, \ldots, X_n)$ given $X_1=x_1, \ldots, X_j=x_j$ is PaU$(\alpha+j, m_j, n-j)$, where $m_j=\max(m, x_1, \ldots, x_j)$.

Proof: (i) is clear from the definition of Pa$U(\alpha, m, n)$. To see (ii), first observe that the joint density of $\Theta, X_1, \ldots, X_n$ is

$$f(\theta, x_1, \ldots, x_n)=\alpha m^\alpha \theta^{-(\alpha+1)} I(\theta>m)\theta^{-n} I((x_1, \ldots, x_n)\in[0, \theta]^n),$$

so the joint density of $(\Theta, X_2, \ldots, X_n)$ given $X_1=x_1$, $f_{x_1}(\theta, x_2, \ldots, x_n)$, is

$$\alpha m^\alpha \theta^{-(\alpha+2)} \theta^{1-n} I(\theta>m') I(x_2, \ldots, x_n)\in[0, \theta]^{n-1}) \frac{\alpha+1}{\alpha}\cdot\frac{(m')^{\alpha+1}}{m^\alpha} \tag{2.6}$$

where $m'=\max(x_1, m)$. But (2.6) is the density of Pa$U(\alpha+1, m', n-1)$. Now (ii) follows easily by induction. □

The next proposition is a result of Ferguson inspired by ideas of Stewart; the present proof is partly a simplification and partly of formalization of his argument, filling a small gap. (The argument that one can work with unconditional expectations in the backward induction is not given in [F]. It appears, in different form, in [BG]).

Proposition 2.4: Given $\varepsilon > 0$ *there exists* $\alpha > 0$ *(close to 0) so that if* $(\Theta, X_1, \dots, X_n)$ *is* $PaU(\alpha, 1, n)$, *then*

$$P(X_t = M_n) \le \varphi_n + \varepsilon \text{ for all } t \in \mathscr{T}_r(X).$$

Proof: Let $W_i = I(X_i = M_n)$, and $W_i^* = I(X_i = M_n^*)$ where $M_i^* = \max(1, M_i)$, $(i = 1, \dots, n)$. Then $W_i^* = W_i I(\max_{i \le n} X_i \ge 1)$. Let $\mathscr{F}_i$ denote the σ-algebra generated by $X_1, \dots, X_i$, and let

$$Y_i = E(W_i^* \mid \mathscr{F}_i).$$

Then $Y_i = P(M_i^* = M_n \mid \mathscr{F}_i)$ on $\{X_i = M_i^*\}$, and $= 0$ otherwise. Note that m is a scale parameter for the Pareto distribution, that is Θ is $\mathrm{Pa}(\alpha, m)$ if and only if Θ/m is $\mathrm{Pa}(\alpha, 1)$, and hence m is also a scale parameter for the $\mathrm{Pa}U(\alpha, m, n)$ distribution.

Fix $\varepsilon > 0$, and let $(\Theta, X_1, \dots, X_n)$ be $\mathrm{Pa}U(\alpha, 1, n)$ where $\alpha > 0$ is chosen so small that

$$P(X_1 < 1) \le \varepsilon/2. \tag{2.7}$$

By Lemma 2.3 (ii), the conditional dsitribution of $(\Theta, X_{j+1}, \dots, X_n)$ given $X_1, \dots, X_j$ depends only on M_j^*, and, by scale-invariance the distribution of

$$(\Theta, X_{j+1}, \dots, X_n)/m_j$$

given $M_j^* = m_j$ is $\mathrm{Pa}U(\alpha + j, 1, n - j)$. Hence,

$$(\Theta, X_{j+1}, \dots, X_n)/M_j^* \text{ is independent of } X_1, \dots, X_j.$$

Thus $P(M_j^* = M_n^* \mid \mathscr{F}_j)$ is constant, and, of course, $= P(M_j^* = M_n^*)$. Letting $c(j) = P(M_j^* = M_n^*)$, $1 \le j \le n$, it is easy to see that

$$0 < c(1) < c(2) < \dots < c(n) = 1.$$

Next, define $t^* \in \mathscr{T}(X)$ by backward induction as follows. Let $t^*(n) = n$, and if $t^*(j+1)$ has been defined for some $j < n$, set

$$t^*(j) = \begin{cases} j & \text{if } Y_j \ge E(Y_{t^*(j+1)} \mid \mathscr{F}_j) \\ t^*(j+1) & \text{otherwise.} \end{cases}$$

Then $t^*(1)$ is the stopping time in $\mathscr{T}(X)$ which maximizes $E(Y_t)=E(W_t^*)$. Let $\gamma_j=E(Y_{t^*(j)} \mid \mathscr{F}_{j-1})$, $j=1,\ldots,n$, (where $\mathscr{F}_0$ is the trivial σ-algebra). By (2.8) $\gamma_n=P(X_n\geq M_{n-1}^* \mid \mathscr{F}_{n-1})$ is constant, and since $(\Theta, X_1, \ldots, X_n)$ is PaU, $\gamma_n>0$.

Assume it has been shown that all γ_j with $j\geq i+1$ are constant. Then $\gamma_j=E(Y_{t^*(j)})$ for $j\geq i+1$, and hence $0<\gamma_n\leq\gamma_{n-1}\leq\ldots\leq\gamma_{i+1}$.

If $c(i)<\gamma_{i+1}$, then Y_i assumes only values smaller than γ_{i+1} and hence $t^*(i)=t^*(i+1)$ and $\gamma_i=\gamma_{i+1}$ is constant. On the other hand, if $c(i)\geq\gamma_{i+1}$, then $t^*(i)=i$ on $\{X_i=M_i^*\}$ and $t^*(i)=t^*(i+1)$ on the complement of this set, in which case

$$\gamma_i=E(c(i)I(\{X_i=M_i^*\})+\gamma_{i+1}I(\{X_i<M_i^*\}) \mid \mathscr{F}_{i-1}).$$

But $\{X_i=M_i^*\}=\{X_i>M_{i-1}^*\}$ a.s., so since X_i/M_{i-1}^* is independent of $(X_1, \ldots, X_{i-1})$ and since $\{X_i<M_i^*\}=\{X_i=M_i^*\}^c$, it follows that γ_i is constant. This implies that all γ_i are constant.

The monotonicity $c(1)<c(2)<\ldots<c(n)$ and $\gamma_1\geq\gamma_2\geq\ldots\geq\gamma_n$ together imply that there exists an r such that

$$t^*(1)=\min\{n, \min\{i\geq r: X_i=M_i^*\}\}.$$

Define $t_r\in\mathscr{T}(R)$ by

$$t_r=\min\{n, \min\{i\geq r: X_i=M_i\}\}=\min\{n, \min\{i\geq r: R_{i,i}=1\}\}.$$

Since $P(X_i=X_j$ for some $i\neq j)=0$, it follows by (2.4) and the definition of φ_n that $E(W_{t_r})\leq\varphi_n$. Since t_1^* is optimal for the process (W_i^*), $E(W_{t^*(1)}^*)\geq E(W_t^*)$ for all $t\in\mathscr{T}(X)$. On $\{X_1\geq 1\}$, t_r agrees with $t^*(1)$ and W_i agrees with W_i^* for all i. Therefore, for every $t\in\mathscr{T}(X)$, it follows by (2.7) that

$$\begin{aligned} P(X_t=M_n)=E(W_t)&\geq E(W_t^*)+\varepsilon/2\leq E(W_{t^*(1)}^*)+\varepsilon/2\\ &\leq E(W_{t^*(1)}I(X_1\geq 1))+\varepsilon=E(W_{t_r})+\varepsilon\\ &\leq\varphi_n+\varepsilon.\end{aligned}$$

Since, for a given sequence $X_1, \ldots, X_n$, there is, for any $t'\in\mathscr{T}_r(X)$, a $t\in\mathscr{T}(X)$ with $P(X_t=M_n)\geq P(X_{t'}=M_n)$, this completes the proof. □

The next theorem generalizes Ferguson's [F] result mainly in that ties (and randomized stopping times) are allowed:

Theorem 2.5:

$$\inf_{X\in\mathscr{E}}\sup_{t\in\mathscr{T}_r(X)} P(X_t=M_n)=\varphi_n=\sup_{t\in\mathscr{T}_r(X)}\inf_{X\in\mathscr{E}} P(X_t=M_n).$$

Proof: As $\mathscr{T}_r(X)\supset\mathscr{T}_r(R)$ both the left-hand side and the right-hand side are $\geq\varphi_n$ by Proposition 2.1. Proposition 2.4 shows that the left-hand side is $\leq\varphi_n$. A fortiori the right-hand side is $\leq\varphi_n$. □

3 Random Number of Observations

We now turn to the situation in which the number N of observable random variables X_i can be random, and the observer does not know the distribution of N. N is assumed bounded with a known upper bound n.

In a game theoretic description, Player I chooses a random number N with $1 \le N \le n$, and given $N=j$ he chooses real-valued exchangeable random variables $X_1, \ldots, X_j$. Player II can look at the values of $X_1, X_2, \ldots$ successively until he decides to stop (or until he notices that there are no further observations available). If he stops at time $t=i$, his reward is

$$V_i = I(N \ge i) I(X_i = M_N)$$

where $M_N = \max\{X_1, \ldots, X_N\}$. On $\{N<i\}$, X_i need not be defined, yet $V_i = 0$ is well-defined on this set.

Let $\mathscr{F}_i$ denote the σ-algebra generated by the sets $\{N \ge j\} \cap \{X_j \le u\}$, $j \le i$, $u \in \mathbb{R}$, and $\mathscr{G}_i$ the σ-algebra generated by the sets $\{N \ge m\} \cap \{R_{j,m} = \ell\}$, $j \le m \le i$, $1 \le \ell \le m$. Using these σ-algebras, $\mathscr{T}(X)$, $\mathscr{T}_r(X)$, $\mathscr{T}(R)$, and $\mathscr{T}_r(R)$ are defined as above.

Any sequence $(q_1, \ldots, q_n)$ with $q_i \in [0, 1]$ $(1 \le i \le n)$ determines a $t \in \mathscr{T}_r(R)$ as follows. For $1 \le i \le n-1$, t stops at time i if t did not stop earlier, $R_{i,i} = 1$ and $U_i \ge q_i$. Let $\mathscr{T}_r^*(R)$ denote the class of randomized stopping rules of this form. Identifying the vector and the stopping time determined by it, we can write $t = (q_1, ,\ldots q_n)$. Let $\mathscr{E}^*$ denote the class of all random vectors $(N, X) = (N, X_1, \ldots, X_N)$ as above without ties $(P(N \le j, X_i = X_j$ for some $i<j) = 0)$.

We now recall the main results of [HK] needed below. Let $\alpha_1 = 1$, $\alpha_2 = 1/2$, and for $n > 2$

$$\alpha_n = \frac{s_{n-1} - s_{k_n - 1}}{(n-k_n)/k_n + (s_{n-1} - s_{k_n - 1}) s_{k_n}}$$

with k_n and s_j as in (2.2). Let $t_n^* = (q_1^*, \ldots, q_n^*) \in \mathscr{T}_r^*(R)$ be defined by

$$q_j^* = \begin{cases} (\alpha_n^{-1} - s_{j-1})^{-1} & \text{for } j = 1,, \ldots, k_n \\ 1 & \text{for } k_n < j \le n. \end{cases}$$

Let $P_n^* = (p_1^*, \ldots, p_n^*)$ be the probability vector with

$$p_j^* = \begin{cases} \alpha_n (j+n)^{-1} & \text{for } j < k_n \\ \alpha_n (1 - (s_{n-1} - s_{k_n - 1})^{-1}) & \text{for } j = k_n \\ 0 & \text{for } k_n < j < n. \end{cases}$$

Theorem B in [HK] asserts: $E(V_{t_n^*}) \ge \alpha_n$ for all $(N, X) \in \mathscr{E}^*$. Theorem C in [HK] asserts: If $(N, X) \in \mathscr{E}^*$ and $P(N=j) = p_j^*$ $(1 \le j \le n)$ then $E(V_t) \le \alpha_n$ for all $t \in \mathscr{T}_r^*(R)$.

Note that Theorem B extends to the case with ties by the device of Proposition 2.1. Clearly, Theorem C does not extend to distributions with ties. One might have $X_1 = X_2 = \ldots = X_N$. Then $t \equiv 1$ is a stopping time with $E(V_t) = 1$.

We can now state the principal result of this paper.

Theorem 3.1: Assume $P(N=j) = p_j^*$ $(j=1, \ldots, n)$. *Let* $(\Theta, X_1, \ldots, X_n)$ *be* $PaU(\alpha, 1, n)$ *and independent of* N. *For any* $\varepsilon > 0$, *if* $\alpha > 0$ *is sufficiently small then*

$$E(V_t) \leq \alpha_n + \varepsilon \text{ for all } t \in \mathscr{T}_r(X). \tag{3.1}$$

Remark: It is convenient to define $X_1, \ldots, X_n$ on all of Ω, although only $X_1, \ldots, X_N$ can be observed.

Before we begin with the proof, let us record two consequences of Theorem 3.1.

Corollary 3.2: The value of the above game exists and equals α_n.

Proof: For any $\varepsilon > 0$, Player I can make sure that $E(V_t) \leq \alpha_n + \varepsilon$ by choosing (N, X) as in Theorem 3.1. On the other hand, Theorem B and its extension to the case with ties imply that Player II can win at least with probability α_n. □

The next corollary strengthens Theorem C since $\mathscr{T}_r^*(R) \subset \mathscr{T}_r(R)$.

Corollary 3.3: If $(N, X) \in \mathscr{E}^*$ *satisfies* $P(N=j) = p_j^*$, $1 \leq j \leq n$, *then* $E(V_t) \leq \alpha_n$ *for all* $t \in \mathscr{T}_r(R)$.

Proof: Given $\{N=j\}$ the distribution of the ranks does not depend on the specific distribution of $(X_1, \ldots, X_j)$ as long as there are no ties. □

Proof of Theorem 3.1: Let $M_i^* = \max(M_i, 1)$ and

$$V_i^* = I(X_i = M_N^*) I(N \geq i).$$

We want to show that for sufficiently small $\alpha > 0$

$$E(V_t^*) \leq \alpha_n + \varepsilon/2 \text{ for all } t \in \mathscr{T}_r(X). \tag{3.2}$$

If $\alpha > 0$ is small, (2.7) holds, and (3.1) follows from (3.2). Let $t \in \mathscr{T}_r(X)$ be given. Define $(q_1, \ldots, q_n)$ by (2.1), and set

$$t_q = \inf\{m \geq 1 : U_m \leq q_m\}.$$

By [PS], t and t_q have the same conditional distribution given $\mathscr{F}_n$. (As the sequence is finite, we can replace $\mathscr{F}_\infty$ by $\mathscr{F}_n$.) Hence

$$E(V_t^*) = E(V_{t_q}^*),$$

and we can assume $t=t_q$. We can also assume $q_n\equiv 1$ and for $i\leq n-1$ $\{q_i>0\}\subset\{X_i=M_i^*\}\cap\{N\geq i\}$, since no reward is paid after time n, and $V_i^*=0$ on the complement of $\{X_i=M_i^*\}\cap\{N\geq i\}$. The plan is to deduce Theorem 3.1 from Theorem C via some changes of the stopping time.

Let $\mathscr{F}_i'$ denote the σ-algebra generated by $X_1, \ldots, X_i$. On $\{N\geq i\}$, $\mathscr{F}_i'$ coincides with $\mathscr{F}_i$. As q_i is $\mathscr{F}_i$-measurable, there exists a measurable function $h_i: \mathbb{R}^i\to[0, 1]$ with $q_i=h_i(X_1, \ldots, X_i)$ on $\{N\geq i\}$. Now let $q_i'=h_i(X_1, \ldots, X_i)$ on Ω, and

$$t_{q'}=\inf\{m\geq 1: U_m\leq q_m'\}.$$

$t'=t_{q'}$ is a randomized stopping time for $(\mathscr{F}_i')$. As $V_i^*=0$ on $\{N<i\}$, we have $E(V_t^*)=E(V_{t'}^*)$. Note that

$$q_n'\equiv 1 \text{ and } \{q_i'>0\}\subset\{X_i=M_i^*\}\quad (i=1, \ldots, n-1). \tag{3.3}$$

Next, we want to replace t' by a stopping time $\hat{t}=t_{\hat{q}}\in\mathscr{T}_r(\mathscr{F}_i')$ where

$$\hat{q}_i=\rho_i I(X_i=M_i^*)\quad (i=1, \ldots, n-1)$$

with $\rho_i\in[0, 1]$ and $\hat{q}_n\equiv 1$. We shall show that this can be done without changing the expected reward. The sequence $\rho_1, \ldots, \rho_{n-1}$ is determined by $\rho_n=1$ and (again taking $0/0=1$)

$$\rho_1=\frac{E(I(X_1\geq 1)q_1')}{P(X_1\geq 1)},$$
$$\rho_i=\frac{E(q_i' I(t'>i-1)I(X_i\geq M_{i-1}^*))}{E(I(t'>i-1)I(X_i\geq M_{i-1}^*))}\quad (i=2, \ldots, n-1).$$

Lemma 3.4: $P(\hat{t}=j)=P(t'=j)$, $j=1, \ldots, n$.

Proof:

$$\begin{aligned}P(t'=1)&=P(X_1\geq 1, q_1'\geq U_1)=E(I(X_1\geq 1)q_1')\\&=\rho_1 P(X_1\geq 1)=E(I(X_1\geq 1)\hat{q}_1)=P(\hat{t}=1).\end{aligned}$$

Note that $\{X_i\geq M_{i-1}^*\}$ is independent of $\mathscr{F}_{i-1}'$. When the assertion has been proved for $j\leq i-1$, we obtain

$$\begin{aligned}P(t'=i)&=E(I(t'>i-1)q_i' I(X_i\geq M_{i-1}^*))\\&=\rho_i E(I(t'>i-1)I(X_i\geq M_{i-1}^*))\\&=\rho_i P(t'>i-1)P(X_i\geq M_{i-1}^*)\\&=\rho_i P(\hat{t}>i-1)P(X_i\geq M_{i-1}^*)\\&=E(\rho_i I(\hat{t}>i-1, X_i\geq M_{i-1}^*))\\&=P(\hat{t}=i).\end{aligned}$$

□

Lemma 3.5: For $j \leq i$

$$E(I(t'=j, N=i)\, V_j^*) = E(I(\hat{t}=j, N=i)\, V_j^*).$$

Proof: On $\{t'=j\}$, $X_j = M_j^*$. Hence

$$\begin{aligned} E(I(t'=j, N=i)\, V_j^*) &= E(I(t'=j, N=i, M_j^* = M_i^*) \\ &= p_i^* E(I(t'=j, M_j^* = M_i^*)) \\ &= p_i^* E(I(t'=j) E(I(M_j^* = M_i^*) \mid X_1, \ldots, X_j, U_1, \ldots, U_j)). \end{aligned}$$

Now note that $E(I(M_j^* = M_i^*) \mid X_1, \ldots, X_j, U_1, \ldots, U_j)$ is a constant $\rho(j, i)$ by (2.8). The same calculations hold with $\hat{t}$ replacing t'. Hence the lemma follows from Lemma 3.4. □

It now follows that

$$E(V_{\hat{t}}^*) = E(V_{\hat{t}}^*).$$

Now assume that (3.2) fails and $\alpha > 0$ is so small that $P(X_1 < 1) \leq \varepsilon/4$. There exists $t \in \mathscr{T}_r(X)$ with $E(V_t^*) > \alpha_n + \varepsilon/2$. Let $\tilde{t}$ be the stopping time $t_{\tilde{q}}$ with

$$\tilde{q}_i(x_1, \ldots, x_i) = \begin{cases} p_i & \text{if } x_i = \max(x_1, x_2, \ldots, x_i) \\ 0 & \text{else.} \end{cases}$$

On $\{X_1 \geq 1\}$, $\tilde{t}$ coincides with $\hat{t}$ and V_i^* with V_i $(i = 1, \ldots, n)$. Hence $E(V_{\tilde{t}}) \geq \alpha_n + \varepsilon/4$. But $\tilde{t} \in \mathscr{T}_r^*(R)$ and the distribution of (N, X) belongs to $\mathscr{E}^*$, which via Theorem C of [HK] contradicts the assumption that (3.2) fails, thereby completing the proof. □

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